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Cyclicity of non-associative products on D-branes

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Abstract: The non-commutative geometry of deformation quantization appears in string theory through the effect of a B-field background on the dynamics of D-branes in the topological limit. For arbitrary backgrounds, associativity of the star product is lost, but only cyclicity is necessary for a description of the effective action in terms of a generalized product. In previous work we showed that this property indeed emerges for a non-associative product that we extracted from open string amplitudes in curved background fields. In the present note we extend our investigation through second order in a complete derivative expansion. We establish cyclicity with respect to the Born-Infeld measure and find a logarithmic correction that modifies the Kontsevich formula in an arbitrary background satisfying the generalized Maxwell equation. This equation is the physical equivalent of a divergence-free Θ, which is required for cyclicity already in the associative case.

Keywords: D-branes, Non-Commutative Geometry.

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1. Introduction

In a seminal paper, M. Kontsevich [1] gave an explicit formula for the deformation quantization of a Poisson structure on $\mathbb{R}^n$ in terms of a formal power series and established the global existence on arbitrary Poisson manifolds using formal geometry. By definition, a deformation quantization is an associative deformation of the commutative product that is proportional to the Poisson bracket $\{f, g\}_\Theta = \Theta^{\mu\nu}\partial_\mu f \partial_\nu g$ to first order in Θ. A symmetric part of Θ would be a Hochschild cocycle, which can be removed by a gauge transformation [1, 2]. The Jacobi identity of the Poisson bracket is equivalent to a vanishing Schouten bracket $[\Theta, \Theta] = 0$, which, in turn, is necessary for the existence of an associative deformation.

Much work on deformation quantization was stimulated by the observation that non-commutative geometry arises in open string theory [3]–[5]. The case of a constant background B-field was shown to lead to a non-commutative product of functions on the world volume of a D-brane, which turned out to be given by the Moyal-Weyl formula. Cattaneo and Felder [6, 7] then gave a physical derivation of Kontsevich's formula in terms of a path integral quantization of a Poisson sigma model [8], which corresponds to an open string theory in a certain topological limit.

More recently the situation of open strings in curved backgrounds was considered and it was shown that the resulting *non-associative* deformation coincides with Kontsevich's expression at first order of a derivative expansion [9]–[11]. In [10] we argued that a non-vanishing field strength $H = dB$ of the 2-form B-field is incompatible with a topological limit of Einstein's equations. The dependence on the metric $g_{\mu\nu}$ therefore should not be ignored. The background fields actually include a gauge connection 1-form A that lives on the brane in addition to the bulk fields $B_{\mu\nu}$ and $g_{\mu\nu}$. The equations of motion, however, can only depend on the gauge-invariant field strength $H = dB$ and $\mathcal{F} = B + (2\pi\alpha')dA$ of B and A.

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Since D-branes can be embedded at arbitrary codimension we expect that only the variational equation for the gauge field plays a role for the non-commutative dynamics while the bulk field backgrounds B and g should remain unconstrained. The antisymmetric non-commutativity parameter $\Theta^{\mu\nu}$ and the "open string metric" $G^{\mu\nu}$ are related to these fields by the matrix inversion $G + \Theta = (g + \mathcal{F})^{-1}$. The generalized Maxwell equation, $G^{\rho\sigma} D_\rho \mathcal{F}_{\sigma\mu} - \frac{1}{2}\Theta^{\rho\sigma} H_{\rho\sigma}{}^{\lambda}\mathcal{F}_{\lambda\mu} = 0$, which comes from the variation of the Born-Infeld measure $\sqrt{\det(g + \mathcal{F})}$ with respect to the gauge connection A, can thus be recast into the form

$$\partial_\mu \left(\sqrt{\det(g + \mathcal{F})}\Theta^{\mu\nu} \right) = 0 \,. \tag{1.1}$$

For the resulting non-associative product [9, 10], we showed that, to first derivative order,

- the integrated product of two functions reduces to the ordinary product and that
- the integrated expression for the associator of three functions vanishes

up to surface terms for the Born-Infeld measure, if the generalized Maxwell equations (1.1) are imposed on the background gauge field [10]. This property is called cyclicity. It is the purpose of the present note to confirm that cyclicity in the above sense can be extended at least through second order in the derivative expansion.

The topological limit corresponds to the situation where the metric is much smaller than all eigenvalues of $\mathcal{F}$ so that $\Theta \approx \mathcal{F}^{-1}$. A vanishing field strength $H = dB = d\mathcal{F} = 0$ (on the D-brane) thus becomes equivalent to the Poisson condition $[\Theta, \Theta] = 0$ and the Born-Infeld measure reduces to the Liouville measure for the symplectic structure Θ. If we then drop the condition that Θ be invertible and consider arbitrary Poisson structures the Kontsevich formula still defines a deformation quantization, but the natural measure is lost. In that context a measure Ω has to be introduced as an independent object [12]. Notably, Felder and Shoikhet constructed a cyclic (gauge-equivalent) modification of the Kontsevich product for Poisson structures Θ that are divergence-free with respect to a measure Ω, i.e.

$$\int_M \Omega \cdot (f * g) \cdot h = \int_M \Omega \cdot (g * h) \cdot f \tag{1.2}$$

for functions $f, g, h \in C^\infty(\mathcal{M})$ of compact support if $\operatorname{div}_\Omega \Theta = 0$ [13]. Using the identity $g * 1 = 1 * g = g$ this immediately implies the generalized Connes-Flato-Sternheimer conjecture [14]:

$$\int_M \Omega \cdot (f * g) = \int_M \Omega \cdot f \cdot g \,. \tag{1.3}$$

In the context of open string theory, there exists a natural measure regardless of the rank of $\mathcal{F}$ or of Θ, and the divergence condition has the natural interpretation of a generalized Maxwell equation (1.1) if Ω is identified with the Born-Infeld measure $\sqrt{\det(g - \mathcal{F})}\, d^D x$. Moreover, cyclicity (1.2), (1.3) of the deformed product can be preserved, at least through second derivative order, even in the non-associative case. We conjecture that this property can be maintained to all orders, but it may then become necessary to take into account derivative corrections to the Born-Infeld measure [15]–[21].

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In this note we explore the cyclicity property at second derivative order of the background fields. Since a diagrammatic calculation along the lines of [10] would be extremely tedious, we check the consistency of our proposal with an ansatz. We should, of course, reproduce the topological limit, which essentially fixes the product up to gauge equivalence. Since we need the explicit expression for the associator we first include all Kontsevich-type graphs without loops with arbitrary coefficients. Associativity up to terms proportional to the 'Jacobiator' $J = \frac{3}{2}[\Theta, \Theta]$ of the Poisson bracket then fixes all coefficients of the ansatz, except for a contribution to the product that is itself proportional to J. (Obviously such a term is not constrained by associativity, but it can be fixed by a symmetry argument.) Thus we recover the known results of [22, 23] and, since we are working in a derivative expansion, extend them to all orders in the constant part of the non-commutativity parameter.

The main focus will then lie on the verification of the cyclicity property. Using the equations of motion for the background gauge field and the expression for the associator, cyclicity also fixes the gauge part of the product. We thus recover the contribution from a loop diagram in Kontsevich's expansion with the same coefficient that was explicitly calculated in [24]. In addition, we find a new term with a logarithmic derivative of the Born-Infeld measure, which restores cyclicity up to terms with at least three derivatives on the background fields Θ and G.

The paper is organized as follows: section 2 contains a discussion of the physical relevance of the cyclicity property and a brief review of the results of [10]. In section 3 we present the ansatz for the non-commutative product and derive the modifications that are required by cyclicity. We conclude with a discussion of our results. The evaluation of the associator is outlined in the appendix.

2. Physical relevance of the cyclicity property

The requirement of a cyclicity property has shown up on a fundamental level of string theory in several places. In the context of open string field theory it constitutes a necessary prerequisite to be able to write down an action which satisfies the BV master equation [25]. An analogous statement is known for closed string field theory [26] and topological strings [27]. In this section we discuss why cyclicity of a non-commutative product is a desirable property in an effective action arising from open string theory, regardless of the associativity of the product. Our arguments will be based on the lagrangian formalism and the variational principle of a (space-time) quantum field theory and on modular invariance of open string theory on the disk. These considerations are quite general and apply to the full non-commutative product emerging from string theory.

As is well known, the space-time low energy effective action can be obtained by computing string amplitudes; the equations of motion for the string background fields emerge from calculating the conformal anomaly. Both quantities should be related by the variational principle in the low energy effective theory. Turning to the perspective of string theory, the purpose of introducing a non-commutative product on the world-volume of a D-brane is to sum up the effect of the background fields in an elegant way. We expect that both the action and the equation of motion can be expressed in terms of a non-commutative product, which means that the antisymmetric background field Θ should only appear implicitly via the product. What are the implications of such an assumption?

To illustrate these considerations we pick some interaction term, say $\int \Phi \circ (\Phi \circ \Phi)$. Applying the variational principle in order to obtain the equations of motion, we obviously obtain three terms. From $\mathrm{SL}(2,\mathbb{R})$ invariance of disk on-shell correlators and from the fact that the properties of the product should not depend on the on-shell condition of the functions (because of the lack of a proper metric dependence), we can expect that the trace property of the integral holds, i.e. that

$$\int (f \circ g) \circ h = \int f \circ (g \circ h)\,, \tag{2.1}$$

as well as $\int f \circ g = \int g \circ f$. Then our variation takes the form $3\int \delta\Phi \circ (\Phi \circ \Phi)$.[1] To separate the contribution to the equation of motion we still have to remove all derivatives from the variation $\delta\Phi$. Doing this by partial integration would produce an explicit Θ dependence in the equation of motion. Therefore, we infer that as a building block of a field theory the product should obey

$$\int f \circ g = \int f \cdot g\,. \tag{2.2}$$

Sticking to our example, we obtain $3\int \delta\Phi \cdot (\Phi \circ \Phi)$, which gives rise to $\Phi \circ \Phi$ in the equation of motion. Therefore we expect that the low energy field theory obtained from open string theory contains a (generically non-commutative and non-associative) product that satisfies the cyclicity property (2.1) and (2.2).

In order to see how this works at first derivative order, and as a warm-up for the calculation in the next section, we briefly review the non-commutative product found in [10]. It was obtained from the computation of off-shell correlators of an open string sigma model with arbitrary, massless, on-shell background fields apart from the dilaton, which was set to zero. In the bulk these are the background metric g and the antisymmetric B-field and at the boundary it is the gauge field A. The product found in [10] to all orders in the non-commutative parameter Θ and to first derivative order in the background fields is

$$f(x) \circ g(x) = f * g - \frac{1}{12}\Theta^{\mu\rho}\partial_\rho\Theta^{\nu\sigma}\left(\partial_\mu\partial_\nu f * \partial_\sigma g + \partial_\sigma f * \partial_\mu\partial_\nu g\right) + \mathcal{O}\left((\partial\Theta)^2, \partial^2\Theta\right), \tag{2.3}$$

where '$*$' denotes the Moyal contribution to the product,

$$f(x) * g(x) = e^{\frac{i}{2}\Theta^{\mu\nu}(x)\partial_{u^\mu}\partial_{v^\nu}} f(u)\; g(v)|_{u=v=x}\,. \tag{2.4}$$

Although formally the same, this represents a non-associative version of Kontsevich's star product formula since Θ is not assumed to define a Poisson structure. Associativity of (2.3) is violated by terms proportional to the Jacobi identity

$$\begin{aligned}(f \circ g) \circ h - f \circ (g \circ h) \;=\; & \frac{1}{6}\left(\Theta^{\mu\sigma}\partial_\sigma\Theta^{\nu\rho} + \Theta^{\nu\sigma}\partial_\sigma\Theta^{\rho\mu} + \Theta^{\rho\sigma}\partial_\sigma\Theta^{\mu\nu}\right)\left[\partial_\mu f * \partial_\nu g * \partial_\rho h\right] + \\ & + \mathcal{O}(\partial^2),\end{aligned} \tag{2.5}$$

[1]Interactions with higher powers in the fields yield a sum over different positionings of brackets.

where we introduced the abbreviation

$$[f * g * h] = e^{\frac{i}{2}\Theta^{\mu\nu}(x)(\partial_{u^\mu}\partial_{v^\nu}+\partial_{u^\mu}\partial_{w^\nu}+\partial_{v^\mu}\partial_{w^\nu})} f(u)\ g(v)\ h(w)\big|_{u=v=w=x}\,, \tag{2.6}$$

which denotes the Moyal-type triple product with all terms containing derivatives on Θ removed. Imposing the generalized Maxwell equation (1.1), it was shown in [10] that the product (2.3) satisfies the cyclicity relations (2.1) and (2.2) to first derivative order of the background fields. The relevant integration measure is given by the Born-Infeld measure $\int = \int \mathrm{d}^D x \sqrt{\det(g - \mathcal{F})}$, which arises from the vacuum amplitude. This result confirms our general arguments above and motivates us to look at further derivative corrections to the product.

Before we go on to the next section and consider the second derivative order, we want to make a comment concerning the Moyal-type triple product $[f*g*h]$. It differs from both $f*(g*h)$ and $(f*g)*h$. Expressions like (2.6) are useful for the evaluation of derivative expansions, since they automatically keep all orders in the undifferentiated Θ. We should keep in mind, however, that we actually work with a double expansion because already the Moyal-type contributions have to be understood as formal power series. We do not use the conventional $\hbar$ to indicate this fact because our derivative expansion is a formal power series in two variables, controlling the number of Θ's and the number of derivatives acting on them, respectively. Our formulas keep terms of arbitrary order in the first parameter and we drop all terms that are cubic in the second one.

3. The non-associative product at second derivative order

In order to check for the cyclicity of the '$\circ$' product, we first need to evaluate the associator. For this purpose it is sufficient to drop all Hochschild coboundaries, i.e. all terms that can be gauged away by a transformation

$$f \circ g \to \mathcal{D}^{-1}(\mathcal{D}f \circ \mathcal{D}g)\,, \qquad \mathcal{D} = 1 + A^{\mu\nu}\partial_\mu\partial_\nu + \cdots\,, \tag{3.1}$$

where $\mathcal{D}$ is some formally invertible differential operator. In particular, contributions to $f \circ g$ of the form $X^{\mu\nu}\partial_\mu f \partial_\nu g$ with symmetric X can be gauged away with $A^{\mu\nu} = -X^{\mu\nu}$. We thus start with an ansatz for the product that contains expressions with arbitrary coefficients for all Kontsevich-type graphs with two derivatives acting on Θ, as displayed in figure 1.

Abbreviating derivatives acting on f and g with subscripts, we obtain the following contributions to the product from Kontsevich-type graphs [1] that contain structures of

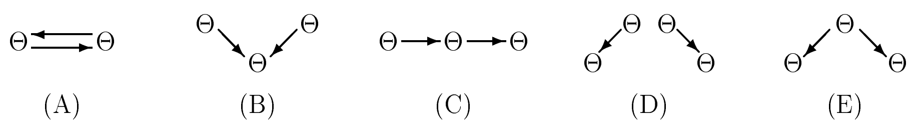

Figure 1: Graphs with two derivatives acting on Θ.

the form in figure $1A$–$1E$:

$$\begin{aligned} f\circ g = f*g &- \frac{1}{12}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\rho}(f_{\mu\nu}*g_\rho + f_\rho * g_{\mu\nu}) + \frac{1}{4}\,\partial_\delta\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta}(A\ f_\mu * g_\nu) - \\ &- \frac{\mathrm{i}}{8}\Theta^{\mu\gamma}\Theta^{\nu\delta}\partial_\gamma\partial_\delta\Theta^{\rho\lambda}(B_1 f_{\mu\nu\rho}*g_\lambda + B_2 f_\lambda * g_{\mu\nu\rho} + B_3 f_{\mu\rho}*g_{\nu\lambda}) - \\ &- \frac{\mathrm{i}}{8}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta}\partial_\delta\Theta^{\rho\lambda}(C_1 f_{\mu\rho}*g_{\nu\lambda} + C_2 f_{\nu\lambda}*g_{\mu\rho} + C_3 f_{\mu\nu\rho}*g_\lambda + C_4 f_\lambda * g_{\mu\nu\rho}) + \\ &+ \frac{1}{16}(\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda})(\Theta^{\nu\delta}\partial_\delta\Theta^{\sigma\tau})(D_1 f_{\mu\rho\nu\sigma}*g_{\lambda\tau} + D_2 f_{\mu\rho\tau}*g_{\lambda\nu\sigma} + D_3 f_{\lambda\tau}*g_{\mu\rho\nu\sigma})\,. \end{aligned} \tag{3.2}$$

There is no contribution from $1E$ because the only possible term $\Theta^{\delta\gamma}\partial_\gamma\Theta^{\mu\nu}\partial_\delta\Theta^{\rho\lambda}f_{\mu\rho}*g_{\nu\lambda}$ vanishes due to symmetry of $\mu\rho$ and $\nu\lambda$ and antisymmetry of Θ. The contribution of graph $1A$ can be gauged away and hence does not contribute to the associator $(f\circ g)\circ h - f\circ(g\circ h)$. Nevertheless, it does contribute to the Kontsevich product with a coefficient $A = 1/6$, which is exactly what we will need for cyclicity.

The evaluation of the associator in the appendix shows that consistency with the topological limit fixes

$$B_1 = -B_2 = \frac{1}{6}\,,\quad B_3 = 0\,,\quad C_2 - C_1 = \frac{1}{3}\,,\quad C_3 = C_4 = 0\,,\quad D_1 = 2D_2 = D_3 = \frac{1}{18}\,. \tag{3.3}$$

The ambiguity $C_1 \to C_1 - C_J$ and $C_2 \to C_2 - C_J$ had to be expected because a contribution of the form $\frac{i}{8}C_J J^{\mu\nu\delta}\partial_\delta\Theta^{\rho\sigma}f_{\mu\rho}*g_{\nu\sigma}$ with

$$J^{\mu\nu\delta} = \frac{3}{2}[\Theta,\Theta]^{\mu\nu\delta} = \Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta} + \Theta^{\nu\gamma}\partial_\gamma\Theta^{\delta\mu} + \Theta^{\delta\gamma}\partial_\gamma\Theta^{\mu\nu} \tag{3.4}$$

generates that shift and vanishes for $J = 0$ (the last term in (3.4) yields a contribution of the form $1E$ that vanishes identically). The Kontsevich formula inherits invariance under the parity transformation exchanging f and g and the sign of Θ from string theory via its topological limit. This symmetry exchanges C_1 with $-C_2$, B_1 with $-B_2$, D_1 with D_3 and leaves all other terms invariant, which implies that the appropriate value is $C_1 = -C_2 = -1/6$. For the sake of generality we will, however, keep the C_2 dependence in the following expressions. The resulting product reads

$$\begin{aligned} f\circ g = f*g &- \frac{1}{12}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\rho}\Big((f_{\mu\nu}*g_\rho + f_\rho*g_{\mu\nu}\Big) + \frac{1}{4}\,\partial_\delta\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta}\ \Big(A\ f_\mu*g_\nu\Big) - \\ &- \frac{\mathrm{i}}{48}\Theta^{\mu\gamma}\Theta^{\nu\delta}\partial_\gamma\partial_\delta\Theta^{\rho\lambda}\Big(f_{\mu\nu\rho}*g_\lambda - f_\lambda*g_{\mu\nu\rho}\Big) - \\ &- \frac{\mathrm{i}}{8}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta}\partial_\delta\Theta^{\rho\lambda}\Big(\Big(C_2-\frac{1}{3}\Big)\ f_{\mu\rho}*g_{\nu\lambda} + C_2\ f_{\nu\lambda}*g_{\mu\rho}\Big) + \\ &+ \frac{1}{2}\frac{1}{12^2}(\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda})(\Theta^{\nu\delta}\partial_\delta\Theta^{\sigma\tau})\Big(f_{\mu\rho\nu\sigma}*g_{\lambda\tau} + 2f_{\mu\rho\tau}*g_{\lambda\nu\sigma} + f_{\lambda\tau}*g_{\mu\rho\nu\sigma}\Big)\,, \end{aligned} \tag{3.5}$$

and for the associator we obtain

$$\begin{aligned} &(f\circ g)\circ h - f\circ(g\circ h) = \\ &= \frac{1}{6}J^{\mu\nu\rho}[f_\mu * g_\nu * h_\rho] + \end{aligned}$$

$$
\begin{aligned}
&+2\Big(\frac{1}{12}\Big)^2(\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda})J^{\nu\sigma\tau}\Big([f_{\mu\rho\nu}*g_\tau*h_{\lambda\sigma}]+[f_{\mu\rho\nu}*g_{\lambda\tau}*h_\sigma]+[f_{\nu\lambda}*g_{\mu\rho\tau}*h_\sigma]+\\
&\qquad+[f_\nu*g_{\mu\rho\tau}*h_{\lambda\sigma}]+2[f_{\mu\nu}*g_{\rho\tau}*h_{\lambda\sigma}]+2[f_{\lambda\nu}*g_{\rho\tau}*h_{\mu\sigma}]+\\
&\qquad+[f_\nu*g_{\lambda\tau}*h_{\mu\rho\sigma}]+[f_{\nu\lambda}*g_\tau*h_{\mu\rho\sigma}]\Big)+\\
&+\frac{\mathrm{i}}{24}J^{\mu\nu\delta}\partial_\delta\Theta^{\rho\lambda}\Big(3C_2[f_{\mu\rho}*g_\nu*h_\lambda]+(3C_2-1)[f_{\mu\rho}*g_\lambda*h_\nu]-\\
&\qquad-3C_2[f_\mu*g_\rho*h_{\nu\lambda}]-(3C_2-1)[f_\rho*g_\mu*h_{\nu\lambda}]+\\
&\qquad+[f_\mu*g_{\nu\rho}*h_\lambda]+[f_\lambda*g_{\nu\rho}*h_\mu]\Big)+\\
&+\frac{\mathrm{i}}{24}\Theta^{\mu\gamma}\partial_\gamma J^{\nu\rho\lambda}\Big([f_{\mu\nu}*g_\rho*h_\lambda]-[f_\rho*g_\lambda*h_{\mu\nu}]\Big),
\end{aligned}
\tag{3.6}
$$

where each term contains the Jacobiator (3.4) as required by consistency with the topological limit.

Associativity up to surface terms. We will now check relation (2.1) for the product (3.5) to second derivative order. To this end we integrate the associator (3.6) with the measure, $\sqrt{\det(g-\mathcal{F})}$, and take the equations of motion for the background fields into account. We will find that lines 3 – 5 of (3.6) vanish by themselves. The same holds for lines 6 and 7. The second line can be pushed to second derivative order and cancels the last two lines.

We start with the easiest piece, the terms proportional to $(\Theta\partial\Theta)J$. In fact, these can all be pushed into the third derivative order by partially integrating one of the derivatives contracted with J, e.g.

$$
\begin{aligned}
&\int d^Dx\sqrt{\det(g-\mathcal{F})}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda}J^{\nu\sigma\tau}[f_{\mu\rho\nu}*g_\tau*h_{\lambda\sigma}]=\\
&=\text{s.t.}-\int d^Dx\partial_\nu\Big(\sqrt{\det(g-\mathcal{F})}J^{\nu\sigma\tau}\Big)\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda}[f_{\mu\rho}*g_\tau*h_{\lambda\sigma}]-\\
&\quad-\int d^Dx(\sqrt{\det(g-\mathcal{F})}J^{\nu\sigma\tau})\partial_\nu(\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda})[f_{\mu\rho}*g_\tau*h_{\lambda\sigma}]-\\
&\quad-\int d^Dx(\sqrt{\det(g-\mathcal{F})}J^{\nu\sigma\tau})\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda}\partial^*_\nu[f_{\mu\rho}*g_\tau*h_{\lambda\sigma}]\\
&\approx 0+\ \mathcal{O}(\partial^3)\,,
\end{aligned}
\tag{3.7}
$$

where the derivative ∂^*_ν acts only on the 'stars' in the product $[f_{\mu\rho}*g_\tau*h_{\lambda\sigma}]$, since J is totally antisymmetric. In a similar way the two lines containing the constant C_2 vanish by partially integrating twice.

The remaining second derivative terms in (3.6) are mixed up with the first derivative order. Therefore let us concentrate on the latter and rewrite it as

$$
\begin{aligned}
&\frac{1}{6}\int d^Dx\sqrt{\det(g-\mathcal{F})}J^{\mu\nu\rho}[f_\mu*g_\nu*h_\rho]=\\
&\quad=\text{s.t.}-\frac{1}{6}\int d^Dx\,\partial_\mu(\sqrt{\det(g-\mathcal{F})}J^{\mu\nu\rho})[f*g_\nu*h_\rho]-\\
&\qquad-\frac{\mathrm{i}}{12}\int d^Dx\sqrt{\det(g-\mathcal{F})}J^{\mu\nu\rho}\partial_\mu\Theta^{\alpha\beta}([f_\alpha*g_{\nu\beta}*h_\rho]+[f_\alpha*g_\nu*h_{\rho\beta}]+[f*g_{\nu\alpha}*h_{\rho\beta}]).
\end{aligned}
\tag{3.8}
$$

The second line of eq. (3.8) can be shown to vanish because of the relation

$$\partial_\mu(\sqrt{\det(g-\mathcal{F})}J^{\mu\nu\rho}) \approx 0\,, \tag{3.9}$$

which holds by way of the equations of motion of the background field (1.1). This can be seen as follows.

Expanding the Jacobiator we find

$$\begin{aligned}\partial_\mu(\sqrt{\det(g-\mathcal{F})}J^{\mu\nu\rho}) \approx\; &(\partial_\mu\sqrt{\det(g-\mathcal{F})})\;(\Theta^{\nu\gamma}\partial_\gamma\Theta^{\rho\mu}+\Theta^{\rho\gamma}\partial_\gamma\Theta^{\mu\nu})+\\ &+\sqrt{\det(g-\mathcal{F})}\;(\partial_\mu\Theta^{\nu\gamma}\partial_\gamma\Theta^{\rho\mu}+\partial_\mu\Theta^{\rho\gamma}\partial_\gamma\Theta^{\mu\nu})+\\ &+\sqrt{\det(g-\mathcal{F})}\;(\Theta^{\nu\gamma}\partial_\mu\partial_\gamma\Theta^{\rho\mu}+\Theta^{\rho\gamma}\partial_\mu\partial_\gamma\Theta^{\mu\nu})\,,\end{aligned} \tag{3.10}$$

where the second line vanishes identically because of the antisymmetry of Θ. Next we exchange the partial derivatives in the last line of (3.10) and use the background field equation (1.1), obtaining

$$\begin{aligned}\partial_\mu(\sqrt{\det(g-\mathcal{F})}J^{\mu\nu\rho})\approx&(\partial_\mu\sqrt{\det(g-\mathcal{F})})\;(\Theta^{\nu\gamma}\partial_\gamma\Theta^{\rho\mu}+\Theta^{\rho\gamma}\partial_\gamma\Theta^{\mu\nu})-\\ &-\sqrt{\det(g-\mathcal{F})}\;\left(\Theta^{\nu\gamma}\partial_\gamma\left(\partial_\mu\sqrt{\det(g-\mathcal{F})}\;\Theta^{\rho\mu}\frac{1}{\sqrt{\det(g-\mathcal{F})}}\right)+\right.\\ &\left.+\Theta^{\rho\gamma}\partial_\gamma\left(\partial_\mu\sqrt{\det(g-\mathcal{F})})\Theta^{\mu\nu}\frac{1}{\sqrt{\det(g-\mathcal{F})}}\right)\right).\end{aligned} \tag{3.11}$$

The terms where the partial derivative in the second and third lines acts on the Θ's cancel the contributions from the first line, while the other terms cancel again, owing to the antisymmetry of the Θ's. Thus we have established our claim (3.9), which shows that from (3.8) only the last line

$$-\frac{\mathrm{i}}{12}\int d^Dx\sqrt{\det(g-\mathcal{F})}J^{\mu\nu\rho}\partial_\mu\Theta^{\alpha\beta}([f_\alpha*g_{\nu\beta}*h_\rho]+[f_\alpha*g_\nu*h_{\rho\beta}]+[f*g_{\nu\alpha}*h_{\rho\beta}]) \tag{3.12}$$

survives. It has to be considered together with other $J\partial\Theta$ contributions in (3.6).

To this end we try to transform the last line of (3.6) into this form. As a first step we rewrite it as

$$\begin{aligned}&\frac{\mathrm{i}}{24}\int d^Dx\sqrt{\det(g-\mathcal{F})}\Theta^{\mu\gamma}\partial_\gamma J^{\nu\rho\lambda}([f_{\mu\nu}*g_\rho*h_\lambda]-[f_\rho*g_\lambda*h_{\mu\nu}])\approx\\ &\quad\approx\frac{1}{48}\int d^Dx\sqrt{\det(g-\mathcal{F})}J^{\nu\rho\lambda}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\alpha\beta}([f_{\mu\nu\alpha}*g_{\rho\beta}*h_\lambda]+[f_{\mu\nu\alpha}*g_\rho*h_{\lambda\beta}]+\\ &\qquad+[f_{\mu\nu}*g_{\rho\alpha}*h_{\lambda\beta}]-[f_{\rho\alpha}*g_{\lambda\beta}*h_{\mu\nu}]-\\ &\qquad-[f_{\rho\alpha}*g_\lambda*h_{\mu\nu\beta}]-[f_\rho*g_{\lambda\alpha}*h_{\mu\nu\beta}])-\\ &\quad-\frac{\mathrm{i}}{24}\int d^Dx\sqrt{\det(g-\mathcal{F})}J^{\nu\rho\lambda}\Theta^{\mu\gamma}([f_{\mu\nu}*g_{\rho\gamma}*h_\lambda]+[f_{\mu\nu}*g_\rho*h_{\lambda\gamma}]-\\ &\qquad-[f_{\rho\gamma}*g_\lambda*h_{\mu\nu}]-[f_\rho*g_{\lambda\gamma}*h_{\mu\nu}])\,,\end{aligned} \tag{3.13}$$

where only the last expression cannot be written as surface term. Note that this is of first derivative order. By partially integrating ∂_ν we obtain

$$\frac{\mathrm{i}}{24}\int d^Dx\sqrt{\det(g-\mathcal{F})}\Theta^{\mu\gamma}\partial_\gamma J^{\nu\rho\lambda}([f_{\mu\nu}*g_\rho*h_\lambda]-[f_\rho*g_\lambda*h_{\mu\nu}])\approx \tag{3.14}$$

$$\approx \frac{\mathrm{i}}{24}\int d^Dx\sqrt{\det(g-\mathcal{F})}J^{\nu\rho\lambda}\partial_\nu\Theta^{\mu\gamma}([f_\mu * g_{\rho\gamma} * h_\lambda] + [f_\mu * g_\rho * h_{\lambda\gamma}] - \\ - [f_{\rho\gamma} * g_\lambda * h_\mu] - [f_\rho * g_{\lambda\gamma} * h_\mu]) - \\ - \frac{1}{48}\int d^Dx\sqrt{\det(g-\mathcal{F})}J^{\nu\rho\lambda}\Theta^{\mu\gamma}\partial_\nu\Theta^{\alpha\beta}([f_{\mu\alpha} * g_{\rho\gamma\beta} * h_\lambda] + [f_{\mu\alpha} * g_{\rho\gamma} * h_{\lambda\beta}] + \\ + [f_\mu * g_{\rho\gamma\alpha} * h_{\lambda\beta}] + [f_{\mu\alpha} * g_{\rho\beta} * h_{\lambda\gamma}] + \\ + [f_{\mu\alpha} * g_\rho * h_{\lambda\gamma\beta}] + [f_\mu * g_{\rho\alpha} * h_{\lambda\gamma\beta}] - \\ - [f_{\rho\gamma\alpha} * g_{\lambda\beta} * h_\mu] - [f_{\rho\gamma\alpha} * g_\lambda * h_{\mu\beta}] - \\ - [f_{\rho\gamma} * g_{\lambda\alpha} * h_{\mu\beta}] - [f_{\rho\alpha} * g_{\lambda\gamma\beta} * h_\mu] - \\ - [f_{\rho\alpha} * g_{\lambda\gamma} * h_{\mu\beta}] - [f_\rho * g_{\lambda\gamma\alpha} * h_{\mu\beta}]) .$$

The last twelve terms in expression (3.14) cancel by partially integrating with respect to ∂_γ, modulo higher derivative orders. Thus we are left with the four terms

$$\frac{\mathrm{i}}{24}\int d^Dx\sqrt{\det(g-\mathcal{F})}J^{\mu\nu\gamma}\partial_\gamma\Theta^{\rho\lambda}([f_\rho*g_{\mu\lambda}*h_\nu]+[f_\rho*g_\mu*h_{\nu\lambda}]-[f_{\mu\lambda}*g_\nu*h_\rho]-[f_\mu*g_{\nu\lambda}*h_\rho]), \tag{3.15}$$

which we have brought into standard index ordering.

Now we are ready to take all remaining terms of (3.6) into account, i.e. expressions (3.12), (3.15) and the seventh line of (3.6). If we rewrite (3.12) as

$$-\frac{\mathrm{i}}{12}\int d^Dx\sqrt{\det(g-\mathcal{F})}J^{\mu\nu\gamma}\partial_\gamma\Theta^{\rho\lambda} \times \\ \times \left([f_\rho * g_{\mu\lambda} * h_\nu] + \frac{1}{2}[f_\rho * g_\mu * h_{\nu\lambda}] - \frac{1}{2}[f_{\mu\lambda} * g_\nu * h_\rho] - [f_\mu * g_{\nu\lambda} * h_\rho]\right), \tag{3.16}$$

and add (3.15) we obtain

$$-\frac{\mathrm{i}}{24}\int d^Dx\sqrt{\det(g-\mathcal{F})}J^{\mu\nu\gamma}\partial_\gamma\Theta^{\rho\lambda}([f_\lambda * g_{\nu\rho} * h_\mu] + [f_\mu * g_{\nu\rho} * h_\lambda]) . \tag{3.17}$$

But this expression cancels exactly the next to last line in (3.6). So we have finally shown that eq. (2.1) is fulfilled in second derivative order, i.e.

$$\int_x (f\circ g)\circ h - f\circ(g\circ h) \;\approx\; \mathcal{O}(\partial^3) . \tag{3.18}$$

In particular, we observe that the constant C_2 remains undetermined.

Ordinary product up to surface terms. We proceed in checking whether the product (3.5) reduces to the ordinary product under the integral. This task is greatly simplified by observing that all terms with third or higher powers in Θ can be pushed to third derivative order. The linear Θ term was already shown to vanish by the background equation (1.1) in [10], so that it remains to consider

$$\int f\circ g \approx \int \Big(f\cdot g - \frac{1}{8}\Theta^{\mu\rho}\Theta^{\nu\sigma} f_{\mu\nu}\cdot g_{\rho\sigma} - \frac{1}{12}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\rho}(f_{\mu\nu}\cdot g_\rho + f_\rho\cdot g_{\mu\nu}) + \\ + \frac{A}{4}\partial_\delta\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta}\; f_\mu\cdot g_\nu\Big) . \tag{3.19}$$

By the usual arguments expression (3.19) can be rewritten as

$$\int \left(f \cdot g + \frac{(6A-1)}{24} \partial_\sigma \Theta^{\mu\rho} \partial_\rho \Theta^{\nu\sigma} f_\mu \cdot g_\nu + \frac{1}{24} \Theta^{\mu\rho} \Theta^{\nu\sigma} \partial_\rho \partial_\sigma (\ln \sqrt{\det(g - \mathcal{F})})\; f_\mu \cdot g_\nu \right). \tag{3.20}$$

Demanding that expression (3.20) becomes the ordinary product of functions requires $A = 1/6$; moreover, it shows that we have forgotten a contribution to the product, which is capable of compensating the last term in (3.20). In fact, we involved only tree level and loop diagrams in the product (3.5), which can be constructed with Θ. In particular, the second term in (3.20) comes from a loop diagram in Kontsevich's expansion. However, the last term is not of this type and arises much in the same manner as the integration measure (cf. [10]). Requiring relation (2.2) therefore determines the explicit dependence of the product on loop contributions, i.e. it fixes the constant A and the factor in front of the logarithmic term.

The product (3.5) therefore becomes

$$\begin{aligned} f \circ g = f * g &- \frac{1}{12} \Theta^{\mu\gamma} \partial_\gamma \Theta^{\nu\rho} (f_{\mu\nu} * g_\rho + f_\rho * g_{\mu\nu}) + \\ &+ \frac{1}{24} \partial_\sigma \Theta^{\mu\rho} \partial_\rho \Theta^{\nu\sigma} f_\mu * g_\nu - \frac{1}{24} \Theta^{\mu\rho} \Theta^{\nu\sigma} \partial_\rho \partial_\sigma (\ln \sqrt{\det(g - \mathcal{F})})\; f_\mu * g_\nu - \\ &- \frac{\mathrm{i}}{8} \Theta^{\mu\gamma} \partial_\gamma \Theta^{\nu\delta} \partial_\delta \Theta^{\rho\lambda} \left(\left(C_2 - \frac{1}{3} \right) f_{\mu\rho} * g_{\nu\lambda} + C_2\; f_{\nu\lambda} * g_{\mu\rho} \right) - \\ &- \frac{\mathrm{i}}{48} \Theta^{\mu\gamma} \Theta^{\nu\delta} \partial_\gamma \partial_\delta \Theta^{\rho\lambda} \; (f_{\mu\nu\rho} * g_\lambda - f_\lambda * g_{\mu\nu\rho}) \frac{1}{2} \frac{1}{12^2} \times \\ &\times (\Theta^{\mu\gamma} \partial_\gamma \Theta^{\rho\lambda})(\Theta^{\nu\delta} \partial_\delta \Theta^{\sigma\tau})(f_{\mu\rho\nu\sigma} * g_{\lambda\tau} + 2 f_{\mu\rho\tau} * g_{\lambda\nu\sigma} + f_{\lambda\tau} * g_{\mu\rho\nu\sigma}). \end{aligned} \tag{3.21}$$

The coefficient for the loop diagram, $\partial\Theta\partial\Theta$, coincides with the result of ref. [24], whereas the term $\ln \sqrt{\det(g - \mathcal{F})}$ represents a new contribution to the product. One may wonder whether this term spoils relation (3.18), but it can be eliminated by a gauge transformation (3.1) and thus has no effect on the associativity.

4. Conclusion

We have constructed a non-associative product that is cyclic with respect to the Born-Infeld measure through second order in the derivative expansion. To this end we have evaluated the associator for the product of three functions on the world-volume of a curved D-brane, whose consistency with the topological limit yields the weights for an infinite number of Kontsevich graphs as a by-product (cf. figure 1). Our product reproduces the Kontsevich formula, including the gauge term, but has an additional contribution with a logarithmic derivative of the measure that may diverge in the topological limit (note that a vanishing divergence of the Poisson structure for some measure is required by cyclicity already in the associative case [13]). In the context of effective low energy actions for open strings in background fields cyclicity, rather than associativity, therefore seems to be the crucial property.

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We conjecture that our results can be extended to arbitrary orders in the derivative expansion, provided that one takes into account corrections from vacuum loops to the Born-Infeld measure, see for instance [15]–[21]. It is well known that certain ambiguities exist in the computation of the renormalized partition function [16], which are related to the scheme dependence of the renormalization procedure. Some of these ambiguities may be fixed by imposing the cyclicity condition using the open string partition function as measure.

Since the non-associativity in the non-topological situation comes from the singularities of the boundary OPEs, which we removed in [10] by subtraction, a proof of our conjecture may require an analysis of the Ward identities and of the A_∞ structure of open string field theory [25, 28].

Recently, string-inspired superspace deformations have attracted a lot of interest [29]–[33]. Such a deformation arises from considering open superstrings in a graviphoton background and can be directly calculated using a covariant quantum description of superstrings with space-time supercoordinates [34]–[36]. Clearly, the starting point for these investigations is constant background fields. A corresponding investigation of non-constant backgrounds is lacking at present. It would be interesting to see how Kontsevich's formula generalizes to a non-commutative product on superspace and whether non-associativity is constrained in these cases by supersymmetry. Furthermore, it would be rewarding to explore the physical aspects of curved brane geometries, such as brane stabilization due to non-trivial background fluxes [37]–[40] in a supersymmetric setting.

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A. Evaluation of the associator

In order to compute the associator for a product that is compatible with the topological limit, we start with the ansatz

$$
\begin{aligned}
f \circ g = f * g &- \frac{1}{12}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\rho}(f_{\mu\nu} * g_\rho + f_\rho * g_{\mu\nu}) - \\
&- \frac{\mathrm{i}}{8}\Theta^{\mu\gamma}\Theta^{\nu\delta}\partial_\gamma\partial_\delta\Theta^{\rho\lambda}(B_1 f_{\mu\nu\rho} * g_\lambda + B_2 f_\lambda * g_{\mu\nu\rho} + B_3 f_{\mu\rho} * g_{\nu\lambda}) - \\
&- \frac{\mathrm{i}}{8}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta}\partial_\delta\Theta^{\rho\lambda}(C_1 f_{\mu\rho} * g_{\nu\lambda} + C_2 f_{\nu\lambda} * g_{\mu\rho} + C_3 f_{\mu\nu\rho} * g_\lambda + C_4 f_\lambda * g_{\mu\nu\rho}) + \\
&+ \frac{1}{16}(\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda})(\Theta^{\nu\delta}\partial_\delta\Theta^{\sigma\tau})(D_1 f_{\mu\rho\nu\sigma} * g_{\lambda\tau} + D_2 f_{\mu\rho\tau} * g_{\lambda\nu\sigma} + D_3 f_{\lambda\tau} * g_{\mu\rho\nu\sigma}),
\end{aligned}
\tag{A.1}
$$

where we used the notation $\partial_\mu f = f_\mu$ for the derivatives acting on the inserted functions. The coefficients B_i, C_i and D_i are arbitrary constants and we dropped the gauge term with coefficient A of (3.2). Compatibility with the case of a Poisson manifold implies that the associator of three functions

$$(f \circ g) \circ h - f \circ (g \circ h) = 0 + \mathcal{O}(J, \partial J) \tag{A.2}$$

only contains terms that are proportional to the Jacobiator $J^{\mu\rho\delta}$ (3.4) or derivatives thereof.

Obviously the terms involving four Θ's do not mix with the other terms in the second derivative order. Inserting the different contibutions to the generalized star product into the associator (A.2), we obtain the following $(\Theta\partial\Theta)^2$ terms from expanding the lowest order part:

$$(f * g) * h - f * (g * h) = \frac{1}{32}(\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda})(\Theta^{\nu\delta}\partial_\delta\Theta^{\sigma\tau})\left([f_{\rho\sigma} * g_{\lambda\tau} * h_{\mu\nu}] - [f_{\mu\nu} * g_{\rho\sigma} * h_{\lambda\tau}]\right), \tag{A.3}$$

where square brackets around the product of three or more functions indicate that there are no derivatives on Θ's contained in theses expressions. Introducing the notation

$$f \circ_1 g = -\frac{1}{12}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\rho}(f_{\mu\nu} * g_\rho + f_\rho * g_{\mu\nu})$$

for the first derivative order we obtain the following terms

$$\begin{aligned}
(f \circ_1 g) * h - f * (g \circ_1 h) &= -\frac{1}{48}(\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda})(\Theta^{\nu\delta}\partial_\delta\Theta^{\sigma\tau}) \times \\
&\times ([f_{\mu\rho\sigma} * g_{\lambda\tau} * h_\nu] + [f_{\lambda\sigma} * g_{\mu\rho\tau} * h_\nu] + [f_\nu * g_{\mu\rho\sigma} * h_{\lambda\tau}] + \\
&\quad + [f_\nu * g_{\lambda\sigma} * h_{\mu\rho\tau}]) \\
(f * g) \circ_1 h - f \circ_1 (g * h) &= -\frac{1}{48}(\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda})(\Theta^{\nu\delta}\partial_\delta\Theta^{\sigma\tau}) \times \\
&\times ([f_{\mu\rho\sigma} * g_\tau * h_{\nu\lambda}] + [f_{\mu\sigma} * g_{\rho\tau} * h_{\nu\lambda}] + [f_{\rho\sigma} * g_{\mu\tau} * h_{\nu\lambda}] + \\
&\quad + [f_\sigma * g_{\mu\rho\tau} * h_{\nu\lambda}] + [f_{\lambda\sigma} * g_\tau * h_{\mu\nu\rho}] + [f_\sigma * g_{\lambda\tau} * h_{\mu\nu\rho}] + \\
&\quad + [f_{\mu\nu\rho} * g_{\lambda\sigma} * h_\tau] + [f_{\mu\nu\rho} * g_\sigma * h_{\lambda\tau}] + [f_{\nu\lambda} * g_{\mu\rho\sigma} * h_\tau] + \\
&\quad + [f_{\nu\lambda} * g_{\mu\sigma} * h_{\rho\tau}] + [f_{\nu\lambda} * g_{\rho\sigma} * h_{\mu\tau}] + [f_{\nu\lambda} * g_\sigma * h_{\mu\rho\tau}]) \\
(f \circ_1 g) \circ_1 h - f \circ_1 (g \circ_1 h) &= \left(\frac{1}{12}\right)^2 (\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda})(\Theta^{\nu\delta}\partial_\delta\Theta^{\sigma\tau}) \times \\
&\times ([f_{\mu\nu\rho\sigma} * g_\tau * h_\lambda] + [f_{\mu\nu\sigma} * g_{\rho\tau} * h_\lambda] + [f_{\rho\nu\sigma} * g_{\mu\tau} * h_\lambda] + \\
&\quad + [f_{\mu\rho\tau} * g_{\nu\sigma} * h_\lambda] + [f_{\mu\tau} * g_{\rho\nu\sigma} * h_\lambda] + [f_{\rho\tau} * g_{\mu\nu\sigma} * h_\lambda] + \\
&\quad + [f_{\nu\lambda\sigma} * g_\tau * h_{\mu\rho}] + [f_{\lambda\tau} * g_{\nu\sigma} * h_{\mu\rho}] - [f_{\mu\rho} * g_{\nu\sigma} * h_{\lambda\tau}] - \\
&\quad - [f_{\mu\rho} * g_\tau * h_{\lambda\nu\sigma}] - [f_\lambda * g_{\mu\nu\sigma} * h_{\rho\tau}] - [f_\lambda * g_{\rho\nu\sigma} * h_{\mu\tau}] - \\
&\quad - [f_\lambda * g_{\nu\sigma} * h_{\mu\rho\tau}] - [f_\lambda * g_{\mu\tau} * h_{\rho\nu\sigma}] - [f_\lambda * g_{\rho\tau} * h_{\mu\nu\sigma}] - \\
&\quad - [f_\lambda * g_\tau * h_{\mu\nu\rho\sigma}]),
\end{aligned} \tag{A.4}$$

where we have used symmetry properties to cancel some contributions. Next, we have to consider the contributions of the second derivative order in (A.1). We use the following notation:

$$f \circ_{(\Theta\partial\Theta)^2} g = -\frac{1}{16}(\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda})(\Theta^{\nu\delta}\partial_\delta\Theta^{\sigma\tau})(D_1 f_{\mu\rho\nu\sigma} * g_{\lambda\tau} + D_2 f_{\mu\rho\tau} * g_{\lambda\nu\sigma} + D_3 f_{\lambda\tau} * g_{\mu\rho\nu\sigma}).$$

The terms arising from this contribution are the only ones where the arbitrary constants D_i enter the calculations. We obtain

$$\begin{aligned}
&(f \circ_{(\Theta\partial\Theta)^2} g) * h - f * (g \circ_{(\Theta\partial\Theta)^2} h) = \\
&= \frac{1}{16}(\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda})(\Theta^{\nu\delta}\partial_\delta\Theta^{\sigma\tau})(D_1[f_{\mu\rho\nu\sigma} * g_{\lambda\tau} * h] + D_2[f_{\mu\rho\tau} * g_{\lambda\nu\sigma} * h] \\
&\qquad + D_3[f_{\lambda\tau} * g_{\mu\rho\nu\sigma} * h] - D_1[f * g_{\mu\rho\nu\sigma} * h_{\lambda\tau}] - \\
&\qquad - D_2[f * g_{\mu\rho\tau} * h_{\lambda\nu\sigma}] - D_3[f * g_{\lambda\tau} * h_{\mu\rho\nu\sigma}])
\end{aligned} \tag{A.5}$$

and

$$\begin{aligned}
&(f * g) \circ_{(\Theta\partial\Theta)^2} h - f \circ_{(\Theta\partial\Theta)^2} (g * h) = \\
&= \frac{1}{16}(\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda})(D_1[(f * g)_{[\mu\rho\nu\sigma]} * h_{\lambda\tau}] + D_2[(f * g)_{[\mu\rho\tau]} * h_{\lambda\nu\sigma}] + \\
&\qquad + D_3[(f * g)_{\lambda\tau} * h_{\mu\rho\nu\sigma}] - D_1[f_{\mu\rho\nu\sigma} * (g * h)_{[\lambda\tau]}] - \\
&\qquad - D_2[f_{\mu\rho\tau} * (g * h)_{[\lambda\nu\sigma]}] - D_3[f_{\lambda\tau} * (g * h)_{[\mu\rho\nu\sigma]}]),
\end{aligned} \tag{A.6}$$

where the indices in square brackets remind us that the derivatives act only on the inserted functions but not on the 'star', since these terms are already of second derivative order. In expanding these expressions we have to be careful, because of the symmetries mentioned above. Putting (A.5) and (A.6) together we find that all terms containing undifferentiated functions cancel. Comparing the result with (A.3) and (A.4), we observe that there are only two contributions with four derivatives acting on f and two contributions with four derivatives acting on g. From the index structure, antisymmetrization makes it clear that terms containing four derivatives acting on the same inserted function can never be absorbed into a term proportional to a Jacobiator. Thus these terms have to cancel

$$(\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda})(\Theta^{\nu\delta}\partial_\delta\Theta^{\sigma\tau})\ [f_{\mu\rho\nu\sigma} * g_\lambda * h_\tau]\ \left(\left(\frac{1}{12}\right)^2 - \frac{D_1}{8}\right) = 0\,,$$

which fixes D_1 to be $D_1 = 1/18$. In the same way we obtain

$$D_3 = D_1 = \frac{1}{18}\,.$$

Now the remaining constant D_2 has to be chosen in such a way that all terms combine to expressions proportional to a Jacobiator. Let us collect all terms of the form $[\partial^3 f * \partial g * \partial^2 h]$. After rearranging the indices, and using the symmetries, we find

$$\left(\frac{1}{9} - D_2\right)\ [f_{\mu\rho\tau} * g_\lambda * h_{\nu\sigma}] + \left(\frac{1}{9} + D_2\right)\ [f_{\mu\rho\sigma} * g_\nu * h_{\lambda\tau}] + \frac{2}{9}\ [f_{\mu\rho\nu} * g_\tau * h_{\lambda\sigma}] + \frac{2}{9}\ [f_{\mu\rho\tau} * g_\sigma * h_{\lambda\nu}]\,.$$

The first term has to vanish, since it cannot, because of the index structure, be expressed as part of a Jacobiator. This fixes the remaining constant to be

$$D_2 = \frac{1}{9}\,.$$

With the same value the remaining three terms are cyclic in $\nu\sigma\tau$ and thus turn the prefactor $(\Theta^{\nu\delta}\partial_\delta\Theta^{\sigma\tau})$ into a full Jacobiator

$$\left(\frac{1}{12}\right)^2 (\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda}) J^{\nu\sigma\tau}\, 2\, [f_{\mu\rho\sigma} * g_\nu * h_{\lambda\tau}]\,, \tag{A.7}$$

where we have written the full expression with the correct numerical prefactor. Repeating this procedure for the other terms gives the following contribution to the associator

$$\begin{aligned}(f\circ g)\circ h - f\circ(g\circ h) \;=\; & 2\left(\frac{1}{12}\right)^2 (\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda}) J^{\nu\sigma\tau}\;\times \\ & \times\Big([f_{\mu\rho\nu} * g_\tau * h_{\lambda\sigma}] + [f_{\mu\rho\nu} * g_{\lambda\tau} * h_\sigma] + [f_{\nu\lambda} * g_{\mu\rho\tau} * h_\sigma] + \\ & \quad + [f_\nu * g_{\mu\rho\tau} * h_{\lambda\sigma}] + 2[f_{\mu\nu} * g_{\rho\tau} * h_{\lambda\sigma}] + 2[f_{\lambda\nu} * g_{\rho\tau} * h_{\mu\sigma}] + \\ & \quad + [f_\nu * g_{\lambda\tau} * h_{\mu\rho\sigma}] + [f_{\nu\lambda} * g_\tau * h_{\mu\rho\sigma}]\Big)\,. \end{aligned} \tag{A.8}$$

Now we turn to the next contributions arising from our ansatz (A.1). Let us consider the part proportional to $\Theta\Theta\partial\partial\Theta$ and collect all terms in the associator that arise from expanding lower order parts of the generalized star product

$$\begin{aligned}(f*g)*h - f*(g*h) \;&=\; \frac{\mathrm{i}}{16}\Theta^{\mu\gamma}\Theta^{\nu\delta}\partial_\gamma\partial_\delta\Theta^{\rho\lambda}\Big([f_{\mu\nu} * g_\rho * h_\lambda] - [f_\rho * g_\lambda * h_{\mu\nu}]\Big) \\ (f\circ_1 g)*h - f*(g\circ_1 h) \;&=\; \frac{\mathrm{i}}{24}\Theta^{\mu\gamma}\Theta^{\nu\delta}\partial_\gamma\partial_\delta\Theta^{\rho\lambda}\Big([f_{\nu\rho} * g_\lambda * h_\mu] + [f_\lambda * g_{\nu\rho} * h_\mu] \\ & \qquad + [f_\mu * g_{\nu\rho} * h_\lambda] + [f_\mu * g_\lambda * h_{\nu\rho}]\Big)\,. \end{aligned} \tag{A.9}$$

From the contributions at second derivative order, terms with undifferentiated functions again do not survive, and we obtain

$$\begin{aligned} & (f\circ_{\partial\partial\Theta} g)*h + (f*g)\circ_{\partial\partial\Theta} h - f*(g\circ_{\partial\partial\Theta} h) - f\circ_{\partial\partial\Theta}(g*h) = \\ & \quad = -\frac{\mathrm{i}}{8}\Theta^{\mu\gamma}\Theta^{\nu\delta}\partial_\gamma\partial_\delta\Theta^{\rho\lambda}\;\times \\ & \quad \times\Big(C_7\ ([f_{\mu\nu} * g_\rho * h_\lambda] + 2\ [f_{\mu\rho} * g_\nu * h_\lambda] + [f_\rho * g_{\mu\nu} * h_\lambda] + +2\ [f_\nu * g_{\mu\rho} * h_\lambda]) - \\ & \qquad - C_8\ ([f_\lambda * g_{\mu\nu} * h_\rho] + 2\ [f_\lambda * g_{\mu\rho} * h_\nu] + [f_\lambda * g_\rho * h_{\mu\nu}] + 2\ [f_\lambda * g_\nu * h_{\mu\rho}]) + \\ & \qquad + C_9\ ([f_\mu * g_\rho * h_{\nu\lambda}] + [f_\rho * g_\mu * h_{\nu\lambda}] - [f_{\mu\rho} * g_\nu * h_\lambda] - [f_{\mu\rho} * g_\lambda * h_\nu])\Big)\,. \end{aligned} \tag{A.10}$$

Collecting the terms with two derivatives acting on the insertion f, we find

$$\begin{aligned}\frac{\mathrm{i}}{16}\Theta^{\mu\gamma}\Theta^{\nu\delta}\partial_\gamma\partial_\delta\Theta^{\rho\lambda}\Big(& (1-2B_1)\ [f_{\mu\nu} * g_\rho * h_\lambda] + \left(\frac{2}{3} + 2B_3\right)\ [f_{\mu\rho} * g_\lambda * h_\nu] + \\ & + (4B_1 - 2B_3)\ [f_{\mu\lambda} * g_\nu * h_\rho]\Big)\,. \end{aligned} \tag{A.11}$$

We observe that the three terms are cyclic in $\nu\rho\lambda$, provided the coefficients are equal; this fixes the constants to be

$$B_1 = \frac{1}{6}\,, \qquad B_3 = 0\,.$$

Going through the same procedure for the terms with two derivatives acting on g, we find

$$\frac{\mathrm{i}}{16}\Theta^{\mu\gamma}\Theta^{\nu\delta}\partial_\gamma\partial_\delta\Theta^{\rho\lambda}\Big((-1-2B_2)\ [f_\rho * g_\lambda * h_{\mu\nu}] - \Big(\frac{2}{3}+2B_3\Big)\ [f_\nu * g_\rho * h_{\mu\lambda}] + \\ + (4B_2-2B_3)[f_\lambda * g_\nu * h_{\mu\rho}]\Big)\,, \tag{A.12}$$

which fixes the constants to

$$B_2 = -\frac{1}{6}\,, \qquad B_3 = 0\,,$$

and is thus compatible with the above values. With these values for B_1 and B_2 the remaining terms cancel and we are left with the following result

$$\frac{\mathrm{i}}{24}\Theta^{\mu\gamma}\Theta^{\nu\delta}\partial_\gamma\partial_\delta\Theta^{\rho\lambda}\Big([f_{\mu\nu} * g_\rho * h_\lambda] + [f_{\mu\rho} * g_\lambda * h_\nu] + [f_{\mu\lambda} * g_\nu * h_\rho] - \\ - [f_\rho * g_\lambda * h_{\mu\nu}] - [f_\nu * g_\rho * h_{\mu\lambda}] - [f_\lambda * g_\nu * h_{\mu\rho}]\Big)\,. \tag{A.13}$$

To turn these expressions into terms proportional to a Jacobiator, we rewrite

$$\Theta^{\mu\gamma}\Theta^{\nu\delta}\partial_\gamma\partial_\delta\Theta^{\rho\lambda} = \Theta^{\mu\gamma}\partial_\gamma(\Theta^{\nu\delta}\partial_\delta\Theta^{\rho\lambda}) - \Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta}\partial_\delta\Theta^{\rho\lambda}\,.$$

Then the first term on the right-hand side gives rise to

$$\frac{\mathrm{i}}{24}\Theta^{\mu\gamma}\partial_\gamma J^{\nu\rho\lambda}([f_{\mu\nu} * g_\rho * h_\lambda] - [f_\rho * g_\lambda * h_{\mu\nu}])\,. \tag{A.14}$$

The remaining terms proportional to $\Theta\partial\Theta\partial\Theta$,

$$-\frac{\mathrm{i}}{24}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta}\partial_\delta\Theta^{\rho\lambda}\Big([f_{\mu\nu} * g_\rho * h_\lambda] + [f_{\mu\rho} * g_\lambda * h_\nu] + [f_{\mu\lambda} * g_\nu * h_\rho] - \\ - [f_\rho * g_\lambda * h_{\mu\nu}] - [f_\nu * g_\rho * h_{\mu\lambda}] - [f_\lambda * g_\nu * h_{\mu\rho}]\Big)\,, \tag{A.15}$$

still have to be considered. To this end we follow the above procedure and collect the terms proportional to $\Theta\partial\Theta\partial\Theta$ arising from lower derivative orders:

$$\begin{aligned}(f \circ_1 g) * h - f * (g \circ_1 h) &= \frac{\mathrm{i}}{24}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta}\partial_\delta\Theta^{\rho\lambda}\Big([f_{\nu\rho} * g_\lambda * h_\mu] + [f_\lambda * g_{\nu\rho} * h_\mu] + \\ &\qquad + [f_\mu * g_{\nu\rho} * h_\lambda] + [f_\mu * g_\lambda * h_{\nu\rho}]\Big) \\ (f * g) \circ_1 h - f \circ_1 (g * h) &= \frac{\mathrm{i}}{24}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta}\partial_\delta\Theta^{\rho\lambda}\Big([f_{\mu\rho} * g_\lambda * h_\nu] + [f_\rho * g_{\mu\lambda} * h_\nu] - \\ &\qquad - [f_\rho * g_\lambda * h_{\mu\nu}] + [f_{\mu\nu} * g_\rho * h_\lambda] - \\ &\qquad - [f_\nu * g_{\mu\rho} * h_\lambda] - [f_\nu * g_\rho * h_{\mu\lambda}]\Big) - \\ &\quad - \frac{\mathrm{i}}{24}\Theta^{\gamma\delta}\partial_\gamma\Theta^{\mu\nu}\partial_\delta\Theta^{\rho\lambda}\Big([f_{\mu\rho} * g_\nu * h_\lambda] + [f_\mu * g_{\nu\rho} * h_\lambda] - \\ &\qquad - [f_\lambda * g_{\mu\rho} * h_\nu] - [f_\lambda * g_\mu * h_{\nu\rho}]\Big)\,.\end{aligned} \tag{A.16}$$

Note the different tensorial structures of these terms. Owing to the symmetries, the second and third terms in the last parenthesis of (A.16) cancel. We rearrange the two terms in

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the second equation of (A.16) proportional to $\Theta^{\gamma\delta}\partial_\gamma\Theta^{\mu\nu}\partial_\delta\Theta^{\rho\lambda}$ by

$$
\begin{aligned}
&-\frac{\mathrm{i}}{24}\Theta^{\gamma\delta}\partial_\gamma\Theta^{\mu\nu}\partial_\delta\Theta^{\rho\lambda}\Big([f_{\mu\rho}*g_\nu*h_\lambda]-[f_\lambda*g_\mu*h_{\nu\rho}]\Big)= \qquad\qquad \text{(A.17)}\\
&\quad=\frac{\mathrm{i}}{24}J^{\mu\nu\delta}\partial_\delta\Theta^{\rho\lambda}\Big([f_{\mu\rho}*g_\nu*h_\lambda]-[f_\lambda*g_\mu*h_{\nu\rho}]\Big)-\\
&\qquad-\frac{\mathrm{i}}{24}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta}\partial_\delta\Theta^{\rho\lambda}\Big([f_{\mu\rho}*g_\nu*h_\lambda]-[f_\lambda*g_\mu*h_{\nu\rho}]-[f_{\nu\rho}*g_\mu*h_\lambda]+[f_\lambda*g_\nu*h_{\mu\rho}]\Big).
\end{aligned}
$$

Putting the pieces of (A.15), (A.16) and (A.17) together, we find:

$$
\begin{aligned}
\frac{\mathrm{i}}{24}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta}\partial_\delta\Theta^{\rho\lambda}\Big(&[f_{\nu\rho}*g_\lambda*h_\mu]+[f_{\nu\rho}*g_\mu*h_\lambda]+[f_\lambda*g_{\nu\rho}*h_\mu]+\\
&+[f_\mu*g_{\nu\rho}*h_\lambda]+[f_\rho*g_{\mu\lambda}*h_\nu]+[f_\nu*g_{\mu\lambda}*h_\rho]\\
&+[f_\mu*g_\lambda*h_{\nu\rho}]+[f_\lambda*g_\mu*h_{\nu\rho}]\Big)\,.
\end{aligned}
$$

The terms with two derivatives acting on the insertion g can be recast to give Jacobiators; thus the contribution of the lower derivative order (A.15) and (A.16) to the associator is given by

$$
\begin{aligned}
&\frac{\mathrm{i}}{24}J^{\mu\nu\delta}\partial_\delta\Theta^{\rho\lambda}\Big([f_{\mu\rho}*g_\nu*h_\lambda]+[f_\lambda*g_\nu*h_{\mu\rho}]+[f_\mu*g_{\nu\rho}*h_\lambda]+[f_\lambda*g_{\nu\rho}*h_\mu]\Big)+ \qquad \text{(A.18)}\\
&+\frac{\mathrm{i}}{24}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta}\partial_\delta\Theta^{\rho\lambda}\Big([f_{\nu\rho}*g_\lambda*h_\mu]+[f_{\nu\rho}*g_\mu*h_\lambda]+[f_\mu*g_\lambda*h_{\nu\rho}]+[f_\lambda*g_\mu*h_{\nu\rho}]\Big)\,.
\end{aligned}
$$

Now we compute the terms that arise from the second derivative order contribution to the $\circ$-product proportional to C_1 and C_2. Again terms involving undifferentiated functions cancel and we obtain

$$
\begin{aligned}
&(f\circ_{\Theta\partial\Theta\partial\Theta}g)*h+(f*g)\circ_{\Theta\partial\Theta\partial\Theta}h-f*(g\circ_{\Theta\partial\Theta\partial\Theta}h)-f\circ_{\Theta\partial\Theta\partial\Theta}(g*h)=\\
&\quad=-\frac{\mathrm{i}}{8}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta}\partial_\delta\Theta^{\rho\lambda}\Big(C_1([f_\mu*g_\rho*h_{\nu\lambda}]+[f_\rho*g_\mu*h_{\nu\lambda}])-\\
&\qquad\qquad+C_2([f_\nu*g_\lambda*h_{\mu\rho}]+[f_\lambda*g_\nu*h_{\mu\rho}])-\\
&\qquad\qquad-C_1([f_{\mu\rho}*g_\nu*h_\lambda]+[f_{\mu\rho}*g_\lambda*h_\nu])-\\
&\qquad\qquad-C_2([f_{\nu\lambda}*g_\mu*h_\rho]+[f_{\nu\lambda}*g_\rho*h_\mu])\Big)\,.
\end{aligned}
$$

In order to arrange the pieces from (A.18) and (A.19) in terms of Jacobiators we have to impose the condition

$$C_2-C_1=\frac{1}{3}$$

on the constants C_1 and C_2. Eventually we obtain

$$
\begin{aligned}
\frac{\mathrm{i}}{24}J^{\mu\nu\delta}\partial_\delta\Theta^{\rho\lambda}\Big(&3C_2[f_{\mu\rho}*g_\nu*h_\lambda]+(3C_2-1)[f_{\mu\rho}*g_\lambda*h_\nu]+[f_\mu*g_{\nu\rho}*h_\lambda]+\\
&+[f_\lambda*g_{\nu\rho}*h_\mu]-3C_2[f_\mu*g_\rho*h_{\nu\lambda}]-(3C_2-1)[f_\rho*g_\mu*h_{\nu\lambda}]\Big)\,. \qquad \text{(A.19)}
\end{aligned}
$$

This already completes the calculations of the associator (A.2), since the terms proportional to C_3 and C_4 cannot be recast into Jacobiators, in view of the symmetrization of $\mu\nu\rho$. Thus the coefficients C_3 and C_4 have to be zero. Hence we obtain the final result

$$\begin{aligned}(f\circ g)\circ h - f\circ(g\circ h) = {} & \frac{1}{6}J^{\mu\nu\rho}[f_\mu * g_\nu * h_\rho] + 2\left(\frac{1}{12}\right)^2 (\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda})J^{\nu\sigma\tau}\times \\ & \times\Big([f_{\mu\rho\nu} * g_\tau * h_{\lambda\sigma}] + [f_{\mu\rho\nu} * g_{\lambda\tau} * h_\sigma] + [f_{\nu\lambda} * g_{\mu\rho\tau} * h_\sigma] \\ & \quad + [f_\nu * g_{\mu\rho\tau} * h_{\lambda\sigma}] + 2[f_{\mu\nu} * g_{\rho\tau} * h_{\lambda\sigma}] + 2[f_{\lambda\nu} * g_{\rho\tau} * h_{\mu\sigma}] \\ & \quad + [f_\nu * g_{\lambda\tau} * h_{\mu\rho\sigma}] + [f_{\nu\lambda} * g_\tau * h_{\mu\rho\sigma}]\Big) + \frac{\mathrm{i}}{24}J^{\mu\nu\delta}\partial_\delta\Theta^{\rho\lambda}\times \\ & \times\Big(3C_2[f_{\mu\rho} * g_\nu * h_\lambda] + (3C_2-1)[f_{\mu\rho} * g_\lambda * h_\nu] - \\ & \quad - 3C_2[f_\mu * g_\rho * h_{\nu\lambda}] - (3C_2-1)[f_\rho * g_\mu * h_{\nu\lambda}] + \\ & \quad + [f_\mu * g_{\nu\rho} * h_\lambda] + [f_\lambda * g_{\nu\rho} * h_\mu]\Big) + \\ & + \frac{\mathrm{i}}{24}\Theta^{\mu\gamma}\partial_\gamma J^{\nu\rho\lambda}\Big([f_{\mu\nu} * g_\rho * h_\lambda] - [f_\rho * g_\lambda * h_{\mu\nu}]\Big)\,. \end{aligned} \tag{A.20}$$

It contains one free parameter, namely C_2. The product (A.1) then reads

$$\begin{aligned} f\circ g = {} & f * g - \frac{1}{12}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\rho}\left(f_{\mu\nu} * g_\rho + f_\rho * g_{\mu\nu}\right) - \\ & - \frac{\mathrm{i}}{8}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta}\partial_\delta\Theta^{\rho\lambda}\left(\left(C_2 - \frac{1}{3}\right) f_{\mu\rho} * g_{\nu\lambda} + C_2\ f_{\nu\lambda} * g_{\mu\rho}\right) - \\ & - \frac{\mathrm{i}}{48}\Theta^{\mu\gamma}\Theta^{\nu\delta}\partial_\gamma\partial_\delta\Theta^{\rho\lambda}\left(f_{\mu\nu\rho} * g_\lambda - f_\lambda * g_{\mu\nu\rho}\right)\frac{1}{2}\frac{1}{12^2}\times \\ & \times(\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda})(\Theta^{\nu\delta}\partial_\delta\Theta^{\sigma\tau})\left(f_{\mu\rho\nu\sigma} * g_{\lambda\tau} + 2f_{\mu\rho\tau} * g_{\lambda\nu\sigma} + f_{\lambda\tau} * g_{\mu\rho\nu\sigma}\right), \end{aligned}$$

and the coefficients coincide with those known from Kontsevich's formula [1, 22, 23].

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